Three Views of a Secret in Covariant Wave Equations

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April 28, 2022

Abstract

The structure of the differential operator for three sets of relativistic wave equations, namely, the Klein-Gordon equation, the Maxwell equations, and the Dirac equation has been shown to be essentially identical using ordinary matrix calculation. The Klein-Gordon equation and the Maxwell equations have been written in the form of conventional Dirac equation with complex matrices, and the Dirac equation has been expressed in the form of Vector Analysis. It is concluded that what makes difference in the appearance of equations is not the differential operator, but the nature of the wave fields to which the operator is applied. The calculations here are based on the matrix manipulation, which is familiar to most of physicists; one do not need esoteric knowledge of modern vector theories to understand the result.

1 Introduction

The Klein-Gordon equation, the Maxwell equations, and the Dirac equation are well-known basic wave equations with manifestly covariant form. The conventional appearance of these three is, however, considerably different. One might feel it would comfortable if it is possible to write three of them in the same form. The Klein-Gordon equation and the Maxwell equations can be expressed more or less in a similar manner. They are written in the framework of Vector Analysis with differential operators such as "$\nabla\cdot$" or "$\nabla\times$". Here in this paper, capitalized words "Vector Analysis" refer to the method developed by Gibbs and Heaviside [1, 2] (the vector analysis we learn in physics departments in the course of electromagnetism) in particular, whereas "vector analysis" in small letters are used to mean mathematical manipulations to handle vectors in general.

In contrast, the differential operator $\gamma^\mu \partial_\mu$ in the conventional Dirac equation looks quite different (γ^μ is the Dirac's gamma matrix). However, as we will see in the present paper, it is essentially identical to the differential operator in the Klein-Gordon or the Maxwell equations. This fact is not new at all. It has been shown by Hesteness in the middle of the last century [7, 8], using an algebraic theory that he christened "Spacetime Algebra". There are several different words for this kind of algebra: Clifford Algebra, Geometric Algebra, or Spacetime Algebra. The meaning of each term is slightly different, however, we do not pay attention to the difference and use the words "Spacetime Algebra" throughout the present paper.

There must be a considerable number of physicists who wish to know why the appearance of basic equations is so different. Therefore, the author of the present paper wonders why it is not well known that the three can be derived in the same manner with Spacetime Algebra, which does not require highly mathematical skill to understand. Perhaps this is partially because we are somewhat conservative by human nature; we do not want to try new tools as long as we are satisfied with old ones, even when a new tool is better. We move on to new tools when we find the cost to learn it is low enough or the benefit is high enough, but do not try when the cost or benefit is unknown.

It should be emphasized that Spacetime Algebra is a truly powerful mathematical tool that can be applied to a wide variety of problems in physics [3]. However, if one can understand what he/she wish to understand without paying the cost of introducing the new tool, we can say it would not be a bad choice to stick to the old tool. Therefore, we try to understand the structure of the wave equations with an old tool, namely matrix manipulation, in the present paper.

Using old style matrix manipulation, we show the three sets of wave equations can be derived from an identical differential operator. This result means it is not the differential operator but the nature of wave fields that make the appearance of equations different. Having elucidated the underlying mechanism of wave equations, we can reexamine the problem in the light of the new tool, namely, Spacetime Algebra. The argument is quite parallel to the one based on matrices, therefore, it may help readers to try the new tool, hopefully.

The n dimensional differential operator we examine here is in the following form in general.

$$\hat{D} = \sum_{\mu}^{n} \hat{\gamma}^{\mu} \partial_{\mu} \,. \tag{1}$$

The terms in the above equation are matrices, which is denoted by hat marks in the present paper; $\hat{\gamma}^{\mu}$ is the Dirac's gamma matrix extended for spaces with general dimensions.

What we wish to show is the above differential operator yields wave equations generally. This operator is often called "Dirac operator", however, we call it "D operator" in the present paper. This is because when we say Dirac operator, it might imply the operator is only for the Dirac equation; actually, it works on all wave equations examined here. The above expression for D operator is the expression in spaces with general dimensions, however, what we are interested in are the cases with $n = 3$ and $n = 4$, which will be scrutinized in the following.

Firstly we delve into the case with $n = 3$, fields in a three-dimensional space. We assume the Euclidian space, therefore, fields are static; static electromagnetic fields are an example. The gamma matrices are reduced to Pauli's sigma matrices. The reason why we start with three-dimensional fields, not full dynamics in the four-dimensional space, is that they are simply easy to handle. We can, of course, start with the four-dimensional Minkowski space from the scratch, however, we have to treat 4×4 complex matrices, which is cumbersome and requires considerable effort to understand the underlying physics. Fields in the three-dimensional Euclidian space, in contrast, can be expressed with 2×2 matrices, which is much easier, and most of the important insights can be obtained with them. We can see the matrix representation has geometrical implication; the sigma matrices correspond to the orthogonal unit vectors.

It is almost straightforward to apply the three-dimensional results to the four-dimensional Minkowski space. We find gamma matrices also have geometrical meaning as unit vectors. It will be shown the same D operator can derive the Klein-Gordon equation, the Maxwell equations, and the Dirac equation. The difference of equations comes not from the operator but from the nature of wave fields. The Klein-Gordon equation is from the line-like vectors and the Maxwell equations are from the plane-like vectors. The Dirac equation is not from the vectors, but from the spinors that represent rotations.

So far, every calculation is done in the matrix base, which is familiar to most of the physicists. However, what we have used is not calculations with actual components of matrices, but their simple algebraic relation only. Therefore, it is possible to introduce an abstract algebra without paying attention to what the actual elements are. This is the starting point of Spacetime Algebra; hopefully, the present paper can help the readers to try Spacetime Algebra with what we have found here.

2 Three-dimensional Static Field

Suppose the following static equations for a three-dimensional vector field $\mathbf{V}$ with a source term s.

$$\nabla \cdot \mathbf{V} = s\,, \quad \nabla \times \mathbf{V} = 0\,. \tag{2}$$

Boldface symbols mean three-dimensional vectors in this paper. Here $\mathbf{V}$ and s are just mathematical fields and do not have to have physical meanings, however, it is easier to understand by the image of electrostatic fields for us physicists. Thus we denote $\mathbf{V} = \mathbf{E}$ and $s = \rho$ in the following

What we wish to do in this section is to reproduce the above static equations using the following Pauli matrices.

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{3}$$

Hat marks indicate matrices in this paper. What we only need for these matrices is to satisfy the following rule.

$$\frac{1}{2}(\hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i) = \hat{1}\,. \tag{4}$$

The matrix $\hat{1}$ in the right hand side is the unit matrix.

Suppose we combine the matrices, differential operators, and field components to make something look like field equations in a *ad hoc* manner without paying attention to their meaning. There can be several choices, and most of them are meaningless. However, if we can find one that gives the result same as (2), it is something. Actually we do not have to try every possible combination here because we know the following correct one with hindsight.

Firstly, let us write the D operator as

$$\hat{D} = \hat{\sigma}_1 \partial_x + \hat{\sigma}_2 \partial_y + \hat{\sigma}_3 \partial_z \,. \tag{5}$$

The vector field may be written as

$$\hat{\mathbf{E}} = \hat{\sigma}_1 E_x + \hat{\sigma}_2 E_y + \hat{\sigma}_3 E_z \,. \tag{6}$$

We can write from (4) the product of two Pauli matrices as

$$\hat{\sigma}_i \hat{\sigma}_j = \begin{cases} \hat{1} & \text{if} \quad i = j \\ -\hat{\sigma}_j \hat{\sigma}_i & \text{if} \quad i \neq j \end{cases} \,. \tag{7}$$

Then we have

$$\begin{aligned} \hat{D}\hat{\mathbf{E}} =& \hat{1}(\partial_x E_x + \partial_y E_y + \partial_z E_z) \\ &+ \hat{\sigma}_z \hat{\sigma}_y (\partial_z E_y - \partial_y E_z) + \hat{\sigma}_z \hat{\sigma}_x (\partial_z E_x - \partial_x E_z) + \hat{\sigma}_x \hat{\sigma}_y (\partial_x E_y - \partial_y E_x) \end{aligned} \tag{8}$$

It is known Pauli matrices with different indices are linearly independent of each other, as well as independent of the unit matrix. Therefore, the above expression has four independent components: one scalar (unit matrix) and three terms with $\hat{\sigma}_i \hat{\sigma}_j$. We can write the field equation as

$$\hat{D}\hat{\mathbf{E}} = \hat{1}\rho \,, \tag{9}$$

which means the scalar term, or unit matrix term, (first line of 8) is ρ and the $\hat{\sigma}_i \hat{\sigma}_j$ terms (second line) are zero. Comparing each component, we understand that the above equation is identical to (2).

Now that we have successfully obtained the equation of the electrostatic field with Pauli matrices, we can scrutinize its meaning. Observing (5) and (6) yield correct answer, we can infer $\hat{\sigma}_i$ represents the unit vector in the x_i direction. This is the meaning of sigma matrices, and we can interpret the above matrix calculation as geometrical vector equations.

The terms with $\hat{\sigma}_i \hat{\sigma}_j \quad (i \neq j)$ correspond to rotation in Vector Analysis: $\nabla \times \mathbf{E}$. This means $\mathbf{E}$ is subject to line integral in the Stokes theorem, in other words, $\mathbf{E}$ is a line-like object, or 1-form in the vocabulary of Cartan's Differential Form [5]. Therefore, we can say the vector equations (6) is obtained by applying D operator on a line-like object. The scalar term ($i = j$) is, in contrast, subject to surface integration, but this term is actually electric flux $\mathbf{D}$, and expressed with the Hodge operator as $\mathbf{D} = {}^*\mathbf{E}$ in the theory of Differential Form; this is is called 3-form.

Now we move on to another kind of vector field $\mathbf{U}$ that is expressed as

$$\nabla \cdot \mathbf{U} = 0\,, \quad \nabla \times \mathbf{U} = \mathbf{S}\,. \tag{10}$$

Again it is better to imagine this field by a field know, and thus we write $\mathbf{U} = \mathbf{B}$ and $\mathbf{S} = \mathbf{J}$ in association with a static magnetic field. The calculation goes almost the same as the electric field case; one difference is that $\mathbf{B}$ is subject to surface integral, therefore $\mathbf{B}$ is a plane-like object; it is called 2-form in the theory of Differential Forms.

$$\hat{\mathbf{B}} = \hat{\sigma}_2\hat{\sigma}_3 B_x + \hat{\sigma}_3\hat{\sigma}_1 B_y + \hat{\sigma}_1\hat{\sigma}_1 B_z\,. \tag{11}$$

Note that each unit plane is spanned by two unit vectors $\hat{\sigma}_i\hat{\sigma}_j$, and the number of planes is three: yz, zx, xy. This is a special situation of a three-dimensional space since ${}_3C_1 = {}_3C_2$. Also it is special to have $\hat{\sigma}_1\hat{\sigma}_2 = i\hat{\sigma}_3$ in three-dimensional case, and thus the unit plane vector $\hat{\sigma}_1\hat{\sigma}_2$ should not be confused with the unit line vector $\hat{\sigma}_3$. The result is

$$\hat{D}\hat{\mathbf{B}} = \mathbf{J}\,, \tag{12}$$

The scalar term means $\nabla \cdot \mathbf{B} = 0$, and the vector are (terms with $\hat{\sigma}_i$ in this case) yields $\nabla \times \mathbf{B} = \mathbf{J}$. This time, the scalar term vanishes to ensure the existence of the vector potential. Here we see that the vector equations of (10) are obtained by applying the D operator on a plane-like object $\mathbf{B}$.

Other points are the same as the those for the electric field. Especially, the D operator is exactly identical to (5), which means it *is not* the differential operator that makes the appearance of equations different. It *is* the operand, wave fields in other words, makes difference. This point becomes important in the four-dimensional Minkowski space; this difference causes the difference of the Klein-Gordon equation and the Maxwell equations.

3 Wave Equations in Minkowski Space

Now we are in the position to apply the tactics in the previous section to the three sets of basic wave equations, namely, the Klein-Gordon equation, the Maxwell equations, and the Dirac equation. As in the previous section, the results here are valid for four-dimensional vectors in general, however, we assume specific fields of the wave equations for easier understanding.

The differential operator and vector fields were represented by 2×2 Pauli matrices in the three-dimensional space. However, 2×2 is too "small" to use in four-dimensional spaces, and we need 4×4 complex matrices. This is not that four is the number of the spacetime dimensions, but that we need this size of matrices to accommodate vectors in four-dimensional spaces.

The requirement for these matrices is the following algebraic rule.

$$\frac{1}{2}(\hat{\gamma}^\mu \hat{\gamma}^\nu + \hat{\gamma}^\nu \hat{\gamma}^\mu) = \hat{\eta}^{\mu\nu} \,, \tag{13}$$

where $\hat{\eta}^{\mu\nu}$ is

$$\hat{\eta}^{\mu\nu} = \begin{cases} \hat{1} & \mu = \nu = 0 \\ -\hat{1} & \text{otherwise} \end{cases} . \tag{14}$$

The definition of the Dirac operator is a straightforward extension of (5); just adding another dimension term.

$$\hat{D} = \hat{\gamma}^t \partial_t + \hat{\gamma}^x \partial_x + \hat{\gamma}^y \partial_y + \hat{\gamma}^z \partial_z \tag{15}$$

As in the three-dimensional case, γ^μ represents the unit vector in the μ direction.

3.1 Klein-Gordon Equation

Let us suppose the following real Klein-Gordon equation.

$$(\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2)\phi = m^2 \phi \,. \tag{16}$$

The story goes quite parallel to the case of the three-dimensional fields. We did not start from the electrostatic potential, but start from the electric field itself in the previous section. The potential may be derived as a result of the field equation (9), which contains not only $\nabla \cdot \mathbf{E} = \rho$, but also $\nabla \times \mathbf{E} = 0$ that assures the existence of the potential.

Similarly we start with the following vector field $\mathcal{V}$ instead of the potential ϕ.

$$\mathcal{V} = \hat{\gamma}^t V_t + \hat{\gamma}^x V_x + \hat{\gamma}^y V_y + \hat{\gamma}^z V_z \,. \tag{17}$$

We denote four-dimensional vectors by calligraphic fonts. The above $\mathcal{V}$ is a line-like object, or 1-form, which corresponds to $\mathbf{V}(= \mathbf{E})$ in (4).

Applying D operator on the above field yields

$$\hat{D}\mathcal{V} = \hat{1}(V_{t,t} + V_{x,x} + V_{y,y} + V_{z,z})$$
$$+ \hat{\gamma}^t\hat{\gamma}^x(V_{t,x} - V_{x,t}) + \hat{\gamma}^t\hat{\gamma}^y(V_{t,y} - V_{y,t}) + \hat{\gamma}^t\hat{\gamma}^z(V_{t,z} - V_{z,t}) \quad (18)$$
$$+ \hat{\gamma}^x\hat{\gamma}^y(V_{x,y} - V_{y,x}) + \hat{\gamma}^z\hat{\gamma}^x(V_{z,x} - V_{x,z}) + \hat{\gamma}^x\hat{\gamma}^y(V_{x,y} - V_{y,x})\,. \quad (19)$$

When the second and third terms vanishes, there exist a potential ϕ whose four-dimensional gradient becomes $\mathcal{V}$. The conventional expression of the Klein-Gordon equation starts with the existence of the potential ϕ; in our approach here the potential is the result of the field equation in contrast. With this potential we arrive at the following expression of the Klein-Gordon equation.

$$\hat{D}\mathcal{V} = m^2\phi\,. \quad (20)$$

3.2 Maxwell Equations

The Klein-Gordon equation above was a differential equation for a line-like object (1-form). The electromagnetic fields in the Maxwell equations is a four-dimensional plane-like object, or 2-form, which corresponds to $\mathbf{U}(=\mathbf{B})$ in (10). This difference of line-like or plane-like makes the appearance of two sets of equations considerably different.

Electromagnetic fields are represented by a six component skew tensor in the covariant relativistic theory. This number (six) is the the maximum number of mutually orthogonal planes in a four-dimensional space: ${}_4C_2 = 6$. The three of the six are the the electric field and the other three are the magnetic field. We can write the electromagnetic field $\mathcal{F}$ as

$$\mathcal{F} = \hat{\gamma}^0\hat{\gamma}^x E_x + \hat{\gamma}^0\hat{\gamma}^y E_y + \hat{\gamma}^0\hat{\gamma}^z E_z + \hat{\gamma}^y\hat{\gamma}^z B_x + \hat{\gamma}^z\hat{\gamma}^x B_y + \hat{\gamma}^x\hat{\gamma}^y B_z \quad (21)$$

Applying D operator to the above expression is similar to the calculation for the Klein-Gordon equation. The following are examples of typical terms.

$$\hat{\gamma}^t\partial_t(\hat{\gamma}^t\hat{\gamma}^x E_x) = \hat{\gamma}^x\partial_t E_x\,, \quad \hat{\gamma}^x\partial_x(\hat{\gamma}^t\hat{\gamma}^x E_x) = \hat{\gamma}^t\partial_x E_x\,, \quad (22)$$
$$\hat{\gamma}_y\partial_y(\hat{\gamma}^t\hat{\gamma}^x E_x) = \hat{\gamma}^t\hat{\gamma}^x\partial_y E_x\,, \quad (23)$$

and

$$\hat{\gamma}^t \partial_t(\hat{\gamma}^y \hat{\gamma}^z B_x) = \hat{\omega} \hat{\gamma}^x \partial_x B_x \,, \quad \hat{\gamma}^x \partial_x(\hat{\gamma}^y \hat{\gamma}^z B_x) = \hat{\omega} \hat{\gamma}^t \partial_x B_x \,,$$
$$\hat{\gamma}_y \partial_y(\hat{\gamma}^t \hat{\gamma}^x B_x) = \hat{\omega} \hat{\gamma}^x \partial_y B_x \,. \tag{24}$$

Here we write $\hat{\omega} = \hat{\gamma}^t \hat{\gamma}^x \hat{\gamma}^y \hat{\gamma}^z$. By arranging the terms we arrive at

$$\begin{aligned} \hat{D}\mathcal{F} =& \hat{\gamma}^t \nabla \cdot \mathbf{E} + \hat{\gamma}^i [-\partial_t \mathbf{E} + \nabla \times \mathbf{B}]_i \\ &+ \hat{\omega} \hat{\gamma}^t \nabla \cdot \mathbf{B} + \hat{\omega} \hat{\gamma}^i [\partial_t \mathbf{B} + \nabla \times \mathbf{E}]_i \,, \end{aligned} \tag{25}$$

where $[\ ]_i$ denotes the i-th component of the vector inside the square bracket.

We use the notation of Vector Analysis so that we can understand how the above equation is related to the Maxwell equations. We do not need to specify the matrix form to obtain the above expression; however, it is easier to use the representation with the Dirac bases for the actual calculation, which is briefly sketched in Appendix.

Defining the electric four current $\mathcal{J}$ as

$$\mathcal{J} = \hat{\gamma}^t \rho + \hat{\gamma}^x J_x + \hat{\gamma}^y J_y + \hat{\gamma}^z J_z \,, \tag{26}$$

the Maxwell equations in the matrix form are expressed as

$$\hat{D}\mathcal{F} = \mathcal{J} \,. \tag{27}$$

This expression means the first line of (25) is equal to the four current and the second line vanishes. The fact that second line vanishes assures the existence of the vector potential.

3.3 Dirac Equation

We have rewritten the Klein-Gordon equation and the Maxwell equations in the form of matrix representation. What we wish to do lastly is to write the Dirac equation, which is expressed with matrices usually, in the form of Vector Analysis. Similar results are obtained by Morgan [13] using Spacetime Algebra.

Now we have the matrix representation of the Maxwell equations (27), which are usually written in the form of Vector Analysis. Then we are able to go inversely from the matrix representation to Vector Analysis representation for the Dirac equation if the electromagnetic field $\mathcal{F}$ in (21) can be replaced by the Dirac field ψ somehow.

The problem is that the electromagnetic field matrix in $\mathcal{F}$ (21) has only six parameters, whereas the Dirac field has four complex parameters, eight real parameters equivalently. We need two more mathematical objects to accommodate the remaining two. These two must be a scalar and $\hat{\omega}$ terms, because we do not have other choices.

$$\begin{aligned}\psi =&\alpha_0 + \hat{\gamma}^0\hat{\gamma}^x\varepsilon_x + \hat{\gamma}^0\hat{\gamma}^y\varepsilon_y + \hat{\gamma}^0\hat{\gamma}^z\varepsilon_z \\ &+ \hat{\gamma}^y\hat{\gamma}^z\beta_x + \hat{\gamma}^z\hat{\gamma}^x\beta_y + \hat{\gamma}^x\hat{\gamma}^y\beta_z + \hat{\omega}\alpha_\omega \end{aligned} \tag{28}$$

Here we have chosen the symbols for components so as to associate the electromagnetic fields in Vector Analysis for intuitive understanding, but they have no direct connection to electromagnetic fields.

To obtain $\hat{D}\psi$ is quite similar to the calculation of (25); difference is the additional terms coming from the four-dimensional gradient of α_0 and α_ω. The temporal vector $\gamma^t m\psi$ in the particle rest frame turns out to give the correct answer as the mass term, and the Dirac equation in the form of Vector Analysis becomes

$$\begin{aligned}\partial_t\alpha_0 + \nabla\cdot\boldsymbol{\varepsilon} = -m\alpha_0\,, &\quad \partial_t\boldsymbol{\varepsilon} - \nabla\alpha_\omega - \nabla\times\boldsymbol{\beta} = m\boldsymbol{\varepsilon} \\ \partial_t\alpha_\omega + \nabla\cdot\boldsymbol{\beta} = -m\alpha_\omega\,, &\quad \partial_t\boldsymbol{\beta} - \nabla\alpha_0 + \nabla\times\boldsymbol{\varepsilon} = m\boldsymbol{\beta}\end{aligned}$$

4 Concluding Remarks

We have derived three sets of basic wave equations, the Klein-Gordon equation, the Maxwell equations, and the Dirac equation namely, using the same differential operator: D operator. What the present paper assert is *the forms of basic wave equations look different because the wave fields to handle are different, not because of the differential operator*. This may not be a new finding, however, the author could not find past literature which clarify this point explicitly.

Usually textbooks on the quantum theory asserts Dirac found the way to express the wave equation for fermions with matrices, and this expression summons the spins. It seems spins are the result of the matrix differential operator in this explanation. This contradicts what we have seen in the present paper because the same differential operator can derive the spinless Klein-Gordon field, as well as the Maxwell equations.

Morgan [13] has derived results similar to ours using Spacetime Algebra. He wrote "The distinction between spin $\frac{1}{2}$ and spin1 has become quite blurred; because both the Dirac wave function and the Maxwell wave function satisfy the same differential equation, ...”. He assumes the difference of the differential equation must cause the difference of wave functions. Our standpoint is opposite: the wave functions are different by their nature from the beginning, which makes the appearance of the differential equations different.

The fields in Klein-Gordon equation and the Maxwell equations are vectors; a line-like object for the Klein-Gordon equation and a plane-like object for the Maxwell equations. The Dirac equation looks quite different from the two above, but we have seen the same D operator can derive it and the result can be cast into the Vector Analysis notation. This time the wave field is not a vector, but a spinor, which is a mathematical object that is different from vectors. It is closely related to the rotation rather than translation for vectors. The difference of vectors and spinors is not well scrutinized here in this paper, and there are many yet to study. However, the difference clearly exists and it makes the appearances of wave equations different. Here we just remark the existence difference and dare not delve deeper into the properties of spinors.

The following table summarizes our result.

Equation	Field	Matrix Form	Vector Analysis Form
Klein-Gordon	line	$\hat{D}\mathcal{V} = m\hat{\phi}$	$\nabla^2\phi = m^2\phi$
Maxwell	plane	$\hat{D}\mathcal{F} = \mathcal{J}$	$\partial_t\mathbf{E} + \nabla \times \mathbf{B} = J$, etc.
Dirac	rotation	$\hat{D}\psi = \hat{m}\psi$	$\partial_t\boldsymbol{\varepsilon} + \nabla \cdot \alpha_0 + \nabla \times \boldsymbol{\beta} = m\boldsymbol{\varepsilon}$, etc.

So far, we have avoided to use the knowledge of Spacetime Algebra in the present paper intentionally. What we have used is well known matrix manipulation so that we can see that the results are obtained without some advanced and sophisticated mathematical theory. Looking back at our calculations, however, all we have used here is the algebraic rule of (13), and we did not need the calculations of actual components. This means we can abandon the actual matrices and regard operators and vectors as more conceptual and abstract mathematical objects that only obey the rule of (13).

This is the starting point of the Spacetime Algebra. It starts by defining algebraic elements with manipulation rules similar to that of matrix calculation, without paying attention to what actually they are. Then it is possible construct a rich and versatile algebraic theory, which would be better than the matrix calculation or other mathematical tools.

We have used words "line-like" and "plane-like" for the objects represented by matrices $\hat{\gamma}^\mu$ and $\hat{\gamma}^\mu\hat{\gamma}^\nu$, respectively, which are called 1-form and 2-form in the theory of Differential Forms. They are termed blades with grade one and grade two in Spacetime Algebra. Those who first study Spacetime Algebra, including the author of the present paper in the past, may be taken aback to find there are objects consists of the sum of blades with different grades.

When we apply D operator on a grade one blade (a line-like vector, or 1-form), for instance, the result is the sum of grade zero (point-like, 0-form) and grade two blades (plane-like, or 2-form). This may seem curious because it is a sum of a scalar and vector in Vector Analysis. However, it is understandable when we interpret it using matrix representation (18) because the result is a sum of 4×4 matrices (note that the first line is the 4×4 unit matrix). This makes sense even when we forget matrices and move on to abstract Spacetime Algebra.

The author of the present paper believes Spacetime Algebra is truly powerful and useful tool that more physicist should learn. Hopefully it would become easier to attack it after reading the present paper, since our matrix approach here is logically the same as the one at the entrance of Spacetime Algebra. We dare not provide tutorial introduction of Spacetime Algebra here because there are considerable number of good ones today [6, 4, 3, 9, 10, 12, 11]. It is recommended to proceed these tutorials if readers find this paper interesting.

Appendix: Actual Matrix Calculation of Maxwell Equations

Here in this appendix we explore the actual matrix form for the Maxwell equations. We essentially do not need matrix components for the algebraic arguments in the main text, however, a specific matrix manipulation may be useful to understand the relation to the formulas in Vector Analysis.

Let us start with the following Dirac basis.

$$\hat{\gamma}^0 = \begin{pmatrix} \hat{1} & 0 \\ 0 & -\hat{1} \end{pmatrix}, \qquad \hat{\gamma}^i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}. \tag{29}$$

The above matrices are of 4×4 and "$\hat{1}$" in $\hat{\gamma^0}$ is the 2×2 unit matrix. The matrix to represent electromagnetic fields is obtained by substituting the above expression to (21) as

$$\mathcal{F} = \begin{pmatrix} i\mathcal{B} & \mathcal{E} \\ \mathcal{E} & i\mathcal{B} \end{pmatrix}, \tag{30}$$

where

$$\mathcal{E} = \begin{pmatrix} E_z & E_x - iE_y \\ E_x + iE_y & -E_z \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix} \tag{31}$$

Then we can write the matrix form of the Maxwell equations as

$$\hat{D}\mathcal{F} = \begin{pmatrix} i\partial_t\mathcal{B} - \hat{\sigma}_i\partial_i\mathcal{E} & \partial_t\mathcal{E} - i\hat{\sigma}_i\partial_i\mathcal{B} \\ \partial_t\mathcal{E} - i\hat{\sigma}_i\partial_i\mathcal{B} & -i\partial_t\mathcal{B} - \hat{\sigma}_i\partial_i\mathcal{E} \end{pmatrix} \tag{32}$$

The calculation for $\hat{\sigma}_i\partial_i\mathcal{E}$ and $\hat{\sigma}_i\partial_i\mathcal{B}$ is not identical, but quite close to that in Section 2. We obtain $\nabla\cdot\mathbf{E}$, $\nabla\times\mathbf{E}$, $\nabla\cdot\mathbf{B}$, and $\nabla\times\mathbf{B}$ terms as in (8). and arrive at the vector form of the Maxwell equations.

The representation with Dirac basis is suitable to obtain the vector form of the Maxwell equations with three vectors $\mathbf{E}$ and $\mathbf{B}$ in usual style. This is understandable because the Dirac representation is essentially $3+1$ split of the spacetime.

In contrast, the Weyl representation emphasizes the chirality, which is equivalent to the helicity for electromagnetic fields. When we use the following Weyl basis

$$\hat{\gamma}^0 = \begin{pmatrix} 0 & \hat{1} \\ \hat{1} & 0 \end{pmatrix}, \qquad \hat{\gamma}^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{33}$$

the electromagnetic field becomes

$$\mathcal{F} = \begin{pmatrix} 0 & \mathcal{E} + i\mathcal{B} \\ \mathcal{E} - i\mathcal{B} & 0 \end{pmatrix}, \tag{34}$$

and the field equation is separated in two opposite polarization parts as expected.

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