Tree Views of a Secret in Covariant Wave Equations

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はじめに

- Unit System
  - $c = \hbar = 1$
  - $-1 = 2 = \frac{1}{2} = \pi = \text{some constants} = 1$
- Slide Template by Tasuku Soma (U. Tokyo)

- URL for this study:
  https://scrapbox.io/flagments/Wave_Equations
  twitter.com/gandhara16

Three sets of Basic Equations

Klein-Gordon	Maxwell	Dirac
$\partial_\mu^2 \phi = m^2 \phi$	$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \text{etc.}$	$\gamma^\mu \partial_\mu \psi_D + m \psi_D = 0$

Why look so different?

Geometric Algebra

- Geometric Algebra can show the three have identical differential operator.
- ... but Geometric Algebra is not in the standard arsenal of physicists.
- Here, the same result is obtained with a standard tool, namely, matrix manipulation.

Geometric Algebra

All you need in this talk is:

$$\hat{\gamma}^\mu \hat{\gamma}^\nu + \hat{\gamma}^\nu \hat{\gamma}^\mu = 2\hat{\eta}^{\mu\nu}$$

unit vector = algebraic object = $\hat{\gamma}^\mu$
($\eta^{\mu\nu}$: metric)

3D Vector Equations

Static Electric Field

$$\nabla \cdot \mathbf{E} = \rho \,, \quad \nabla \times \mathbf{E} = 0 \,.$$

Static Magnetic Field

$$\nabla \times \mathbf{B} = \mathbf{J} \,, \quad \nabla \cdot \mathbf{B} = 0.$$

Tactics

Express with Pauli Matrices $\hat{\sigma}_i$.

Pauli Matrices

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(hat mark = matrix)

All we need is: $\frac{1}{2}(\hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i) = \hat{1}\delta_{ij}$

$$\hat{\sigma}_i\hat{\sigma}_j = \begin{cases} \hat{1} & \text{if } i = j \\ -\hat{\sigma}_j\hat{\sigma}_i & \text{if } i \neq j \end{cases}.$$

Electric Field Equation with Pauli Matrices

Dirac Operator

$$\hat{D} = \hat{\sigma}_1 \partial_x + \hat{\sigma}_2 \partial_y + \hat{\sigma}_3 \partial_z \, .$$

Electric Field

$$\hat{\mathbf{E}} = \hat{\sigma}_1 E_x + \hat{\sigma}_2 E_y + \hat{\sigma}_3 E_z \, .$$

Electric Field Equation

$$
\begin{aligned}
\hat{D}\hat{\mathbf{E}} =& (\hat{\sigma}_1\partial_x + \hat{\sigma}_2\partial_y + \hat{\sigma}_3\partial_z)(\hat{\sigma}_1 E_x + \hat{\sigma}_2 E_y + \hat{\sigma}_3 E_z) \\
=& \hat{1}(\partial_x E_x + \partial_y E_y + \partial_z E_z) \\
& + \hat{\sigma}_3\hat{\sigma}_2(\partial_z E_y - \partial_y E_z) + \hat{\sigma}_3\hat{\sigma}_1(\partial_z E_x - \partial_x E_z) \\
& + \hat{\sigma}_1\hat{\sigma}_2(\partial_x E_y - \partial_y E_x)
\end{aligned}
$$

$$
\hat{\sigma}_i\hat{\sigma}_j = \begin{cases} \hat{1} & \text{if } i = j \\ -\hat{\sigma}_j\hat{\sigma}_i & \text{if } i \neq j \end{cases}
$$

$$
\begin{aligned}
\hat{\sigma}_1\partial_x\hat{\sigma}_1 E_X &= \hat{1}\partial_x E_x \\
\hat{\sigma}_1\partial_x\hat{\sigma}_2 E_y &= \hat{\sigma}_1\hat{\sigma}_2\partial_x E_y
\end{aligned}
$$

Vector Notation

$$
\begin{aligned}
\hat{D}\hat{\mathbf{E}} =& (\hat{\sigma}_1\partial_x + \hat{\sigma}_2\partial_y + \hat{\sigma}_3\partial_z)(\hat{\sigma}_1 E_x + \hat{\sigma}_2 E_y + \hat{\sigma}_3 E_z) \\
=& \hat{1}(\partial_x E_x + \partial_y E_y + \partial_z E_z) \\
&+ \hat{\sigma}_3\hat{\sigma}_2(\partial_z E_y - \partial_y E_z) + \hat{\sigma}_3\hat{\sigma}_1(\partial_z E_x - \partial_x E_z) \\
&+ \hat{\sigma}_1\hat{\sigma}_2(\partial_x E_y - \partial_y E_x) \\
=& \hat{1}\rho
\end{aligned}
$$

$$
\hat{D}\hat{\mathbf{E}} = \begin{cases} \nabla \cdot \mathbf{E} = \rho & (i = j) \\ \nabla \times \mathbf{E} = 0 & (i \neq j) \quad \longrightarrow \quad \mathbf{E} = \nabla\phi \end{cases}
$$

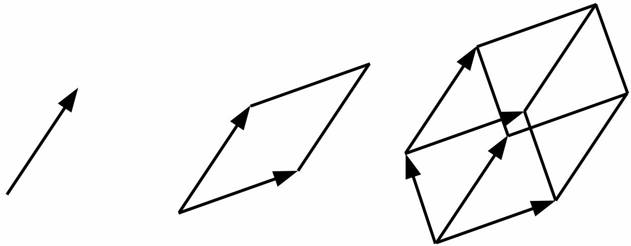
$\hat{\sigma}$ as Unit Vectors

$$\hat{D} = \hat{\sigma}_1\partial_x + \hat{\sigma}_2\partial_y + \hat{\sigma}_3\partial_z$$
$$\hat{\mathbf{E}} = \hat{\sigma}_1 E_x + \hat{\sigma}_2 E_y + \hat{\sigma}_3 E_z$$

$$\begin{aligned}
\hat{D}\hat{\mathbf{E}} =& (\hat{\sigma}_1\partial_x + \hat{\sigma}_2\partial_y + \hat{\sigma}_3\partial_z)(\hat{\sigma}_1 E_x + \hat{\sigma}_2 E_y + \hat{\sigma}_3 E_z) \\
=& \hat{1}(\partial_x E_x + \partial_y E_y + \partial_z E_z) \\
& + \hat{\sigma}_3\hat{\sigma}_2(\partial_z E_y - \partial_y E_z) + \hat{\sigma}_3\hat{\sigma}_1(\partial_z E_x - \partial_x E_z) \\
& + \hat{\sigma}_1\hat{\sigma}_2(\partial_x E_y - \partial_y E_x) \\
\Rightarrow& \nabla \cdot \mathbf{E} + \nabla \times \mathbf{E}
\end{aligned}$$

$$\hat{\sigma}_i \Leftrightarrow \mathbf{e}_i\,, \qquad \hat{\sigma}_i\hat{\sigma}_j \Leftrightarrow \begin{cases} 1 & (i = j) \\ \mathbf{e}_i \times \mathbf{e}_k & (i \neq j) \end{cases}$$

Multi Vectors

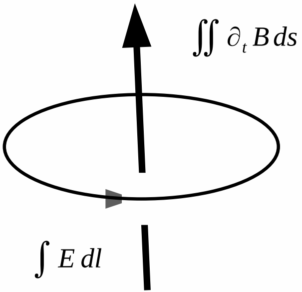


$\hat{\sigma}_i \Leftrightarrow \mathbf{e}_i \quad \hat{\sigma}_i\hat{\sigma}_j \Leftrightarrow \mathbf{e}_i \times \mathbf{e}_j \quad \hat{\sigma}_i\hat{\sigma}_j\hat{\sigma}_k \Leftrightarrow \mathbf{e}_k \cdot (\mathbf{e}_i \times \mathbf{e}_j)$

grade	0	1	2	3
unit	$\hat{1}$	$\hat{\sigma}_i$	$\hat{\sigma}_i\hat{\sigma}_j$	$\hat{\sigma}_i\hat{\sigma}_j\hat{\sigma}_k$
object	point	line	surface	voume
components	1	3	3	1

Magnetic Field with Pauli Matrices

$$\hat{\mathbf{B}} = \hat{\sigma}_2\hat{\sigma}_3 B_x + \hat{\sigma}_3\hat{\sigma}_1 B_y + \hat{\sigma}_1\hat{\sigma}_2 B_z$$



Magnetic Field Equation

Dirac Operator

$$\hat{D} = \hat{\sigma}_1 \partial_x + \hat{\sigma}_2 \partial_y + \hat{\sigma}_3 \partial_z \,.$$

Magnetic Field

$$\hat{\mathbf{B}} = \hat{\sigma}_2 \hat{\sigma}_3 B_x + \hat{\sigma}_3 \hat{\sigma}_1 B_y + \hat{\sigma}_1 \hat{\sigma}_2 B_z$$

$$\begin{aligned}
\hat{D}\hat{\mathbf{B}} =& (\hat{\sigma}_1 \partial_x + \hat{\sigma}_2 \partial_y + \hat{\sigma}_3 \partial_z)(\hat{\sigma}_2 \hat{\sigma}_3 B_x + \hat{\sigma}_3 \hat{\sigma}_1 B_y + \hat{\sigma}_1 \hat{\sigma}_2 B_z) \\
=& \hat{\sigma}_1 (\partial_z B_y - \partial_y B_z) + \hat{\sigma}_2 (\partial_z B_x - \partial_x B_z) \\
& + \hat{\sigma}_3 (\partial_x B_y - \partial_y B_x) \\
& + \hat{\sigma}_1 \hat{\sigma}_2 \hat{\sigma}_3 (\partial_x B_x + \partial_y B_y + \partial_z B_z) \\
\Rightarrow & \nabla \times \mathbf{B} + \nabla \cdot \mathbf{B}
\end{aligned}$$

Electric and Magnetic Fields

$$
\begin{aligned}
\hat{D}\hat{\mathbf{E}} =& (\hat{\sigma}_1\partial_x + \hat{\sigma}_2\partial_y + \hat{\sigma}_3\partial_z)(\hat{\sigma}_1 E_x + \hat{\sigma}_2 E_y + \hat{\sigma}_3 E_z) \\
=& \hat{1}(\partial_x E_x + \partial_y E_y + \partial_z E_z) \\
& + \hat{\sigma}_3\hat{\sigma}_2(\partial_z E_y - \partial_y E_z) + \hat{\sigma}_3\hat{\sigma}_1(\partial_z E_x - \partial_x E_z) \\
& + \hat{\sigma}_1\hat{\sigma}_2(\partial_x E_y - \partial_y E_x) \\
\Rightarrow & \nabla \cdot \mathbf{E} + \nabla \times \mathbf{E}
\end{aligned}
$$

$$
\begin{aligned}
\hat{D}\hat{\mathbf{B}} =& (\hat{\sigma}_1\partial_x + \hat{\sigma}_2\partial_y + \hat{\sigma}_3\partial_z)(\hat{\sigma}_2\hat{\sigma}_3 B_x + \hat{\sigma}_3\hat{\sigma}_1 B_y + \hat{\sigma}_1\hat{\sigma}_2 B_z) \\
=& \hat{\sigma}_1(\partial_z B_y - \partial_y B_z) + \hat{\sigma}_2(\partial_z B_x - \partial_x B_z) \\
& + \hat{\sigma}_3(\partial_x B_y - \partial_y B_x) \\
& + \hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3(\partial_x B_x + \partial_y B_y + \partial_z B_z) \\
\Rightarrow & \nabla \times \mathbf{B} + \nabla \cdot \mathbf{B}
\end{aligned}
$$

Operator and Operand

Electric Field line: $\mathbf{E} = \hat{\sigma}_i E_i$

$$\hat{D}\hat{\mathbf{E}} = (\hat{\sigma}_i D_i)(\hat{\sigma}_j E_j) = \hat{1}\nabla \cdot \mathbf{E} + \hat{\sigma}_i \hat{\sigma}_j \epsilon_{kij} [\nabla \times \mathbf{E}]_k$$

Magnetic Field surface: $\mathbf{B} = \hat{\sigma}_i \hat{\sigma}_j \epsilon_{kij} B_k$

$$\hat{D}\hat{\mathbf{B}} = (\hat{\sigma}_i D_i)(\hat{\sigma}_i \hat{\sigma}_j B_j) = \hat{\sigma}_i [\nabla \times \mathbf{B}]_i + \hat{\sigma}_i \hat{\sigma}_j \hat{\sigma}_k \nabla \cdot \mathbf{B}$$

grade	0	1	2	3
unit	$\hat{1}$	$\hat{\sigma}_i$	$\hat{\sigma}_i \hat{\sigma}_j$	$\hat{\sigma}_i \hat{\sigma}_j \hat{\sigma}_k$
object	point	line	surface	voume
components	1	3	3	1
elec.	$\nabla \cdot \mathbf{E}$	$\mathbf{E}$	$\nabla \times \mathbf{E}$	
mag.		$\nabla \times \mathbf{B}$	$\mathbf{B}$	$\nabla \cdot \mathbf{B}$

4D Vectors

- Unit Vector: $\hat{\sigma}_i \Longrightarrow \hat{\gamma}^\mu$
- Operator: $\hat{D} = \hat{\gamma}^t \partial_t + \hat{\gamma}^x \partial_x + \hat{\gamma}^y \partial_y - + \hat{\gamma}^z \partial_z$

$$\hat{\gamma}^\mu \hat{\gamma}^\nu + \hat{\gamma}^\nu \hat{\gamma}^\mu = 2\hat{\eta}^{\mu\nu}$$

($\hat{\eta}$: metric)

3D $\Longrightarrow$ 4D

grade	0	1	2	3
unit	$\hat{1}$	$\hat{\sigma}_i$	$\hat{\sigma}_i\hat{\sigma}_j$	$\hat{\sigma}_i\hat{\sigma}_j\hat{\sigma}_k$
object	point	line	surface	voume
components	1	3	3	1
elec.	$\nabla \cdot \mathbf{E}$	$\mathbf{E}$	$\nabla \times \mathbf{E}$	
mag.		$\nabla \times \mathbf{B}$	$\mathbf{B}$	$\nabla \cdot \mathbf{B}$

grade	0	1	2	3	4
unit	$\hat{1}$	$\hat{\gamma}^\mu$	$\hat{\gamma}^\mu\hat{\gamma}^\nu$	$\hat{\gamma}^\mu\hat{\gamma}^\nu\hat{\gamma}^\alpha$	$\hat{\gamma}^\mu\hat{\gamma}^\nu\hat{\gamma}^\alpha\hat{\gamma}^\beta$
object	point	line	surface	3-volume	4-voume
components	1	4	6	4	1
line-type	$\hat{\delta}\hat{\mathcal{V}}$	$\hat{\mathcal{V}}$	$\hat{d}\hat{\mathcal{V}}$		
surface-type		$\hat{\delta}\hat{\mathcal{F}}$	$\hat{\mathcal{F}}$	$\hat{d}\hat{\mathcal{F}}$	

Vector Equations: Static 3D EM Fields

- Operator:

$$\hat{D} = \hat{\sigma}_x \partial_x + \hat{\sigma}_y \partial_y + \hat{\sigma}_z \partial_z$$

- Operand:
  - Line $\Rightarrow$ Electric Field

$$\hat{\mathbf{E}} = \hat{\sigma}_1 E_x + \hat{\sigma}_2 E_y + \hat{\sigma}_3 E_z$$

  - Surface $\Rightarrow$ Magnetic Field

$$\hat{\mathbf{B}} = \hat{\sigma}_2 \hat{\sigma}_3 B_x + \hat{\sigma}_3 \hat{\sigma}_1 B_y + \hat{\sigma}_1 \hat{\sigma}_2 B_z$$

Vector Equations: 4D

- Operator:

$$\hat{D} = \hat{\gamma}^t \partial_t + \hat{\gamma}^x \partial_x + \hat{\gamma}^y \partial_y - + \hat{\gamma}^z \partial_z$$

- Operand:
    - Line $\Longrightarrow$ Klein-Gordon

$$\hat{\mathcal{V}} = \hat{\gamma}^t \hat{\mathcal{V}}_t + \hat{\gamma}^x \hat{\mathcal{V}}_x + \hat{\gamma}^y \hat{\mathcal{V}}_y - + \hat{\gamma}^z \hat{\mathcal{V}}_z$$

    - Surface $\Longrightarrow$ Maxwell

$$\begin{aligned} \hat{\mathcal{F}} = & \hat{\gamma}^t \hat{\gamma}^x E_x + \hat{\gamma}^t \hat{\gamma}^y E_x + \hat{\gamma}^t \hat{\gamma}^z E_z \\ & + \hat{\gamma}^y \hat{\gamma}^z B_x + \hat{\gamma}^z \hat{\gamma}^x B_y + \hat{\gamma}^x \hat{\gamma}^y B_z \end{aligned}$$

The Klein-Gordon Equation

$$\hat{\mathcal{V}} = \hat{\gamma}^{\mu}\partial_{\mu}\phi$$

$$\begin{aligned}
\hat{D}\hat{\mathcal{V}} =& \hat{\gamma}^{t}\hat{\gamma}^{x}(V_{t,x} - V_{x,t}) + \hat{\gamma}^{t}\hat{\gamma}^{y}(V_{t,y} - V_{y,t}) \\
&+ \hat{\gamma}^{t}\hat{\gamma}^{z}(V_{t,z} - V_{z,t}) + \hat{\gamma}^{x}\hat{\gamma}^{y}(V_{x,y} - V_{y,x}) \\
&+ \hat{\gamma}^{z}\hat{\gamma}^{x}(V_{z,x} - V_{x,z}) + \hat{\gamma}^{x}\hat{\gamma}^{y}(V_{x,y} - V_{y,x}) \\
=& \hat{1}(V_{t,t} + V_{x,x} + V_{y,y} + V_{z,z}) \\
\equiv& \hat{\delta}\hat{\mathcal{V}} + \hat{d}\hat{\mathcal{V}}
\end{aligned}$$

$$\hat{D}\hat{\mathcal{V}} = \begin{cases} \hat{\delta}\hat{\mathcal{V}} = 0 & \longrightarrow \quad \hat{\mathcal{V}} = \nabla \cdot \phi \\ \hat{d}\hat{\mathcal{V}} = m^2\phi & \end{cases}$$

Electrostatic Field

$$\mathbf{E} = \nabla \cdot \phi$$

$$\begin{aligned}\hat{D}\hat{\mathbf{E}} =& + \hat{\sigma}_3\hat{\sigma}_2(\partial_z E_y - \partial_y E_z) + \hat{\sigma}_3\hat{\sigma}_1(\partial_z E_x - \partial_x E_z) \\ & + \hat{\sigma}_1\hat{\sigma}_2(\partial_x E_y - \partial_y E_x) \\ & + \hat{1}(\partial_x E_x + \partial_y E_y + \partial_z E_z) \\ =& \hat{1}\rho\end{aligned}$$

$$\hat{D}\hat{\mathbf{E}} = \begin{cases}\nabla \times \mathbf{E} = 0 & (i \neq j) \\ \nabla \cdot \mathbf{E} = \rho & (i = j)\end{cases} \longrightarrow \quad \mathbf{E} = \nabla\phi$$

Line-type: The Klein-Gordon Equation

$$\hat{D}\hat{\mathcal{V}} = (\hat{\delta} + \hat{d})\hat{\mathcal{V}} = m^2\phi$$

components	1	4	6	4	1
object	point	line	surface	3-volume	4-voume
line-type	$\hat{\delta}\hat{\mathcal{V}}$	$\hat{\mathcal{V}}$	$\hat{d}\hat{\mathcal{V}}$		
surface-type		$\hat{\delta}\hat{\mathcal{F}}$	$\hat{\mathcal{F}}$	$\hat{d}\hat{\mathcal{F}}$	

$$\hat{\mathcal{V}} = \hat{\gamma}^{\mu}\partial_{\mu}\phi$$

Surface-type: Maxwell Equations

$$\hat{\mathcal{F}} = \hat{\gamma}^{\mu}\hat{\gamma}^{\nu}F_{\mu\nu} : \qquad F_{0i} = E_i , \quad F_{ij} = \epsilon_{kij}B_k$$

$$\hat{\gamma}^t\partial_t(\hat{\gamma}^t\hat{\gamma}^x E_x) = \hat{\gamma}^x\partial_t E_x\,, \quad \hat{\gamma}^x\partial_x(\hat{\gamma}^t\hat{\gamma}^x E_x) = \hat{\gamma}^t\partial_x E_x\,,$$
$$\hat{\gamma}_y\partial_y(\hat{\gamma}^t\hat{\gamma}^x E_x) = \hat{\gamma}^t\hat{\gamma}^x\partial_y E_x\,,$$
$$\hat{\gamma}^t\partial_t(\hat{\gamma}^y\hat{\gamma}^z B_x) = \hat{\omega}\hat{\gamma}^x\partial_x B_x\,, \quad \hat{\gamma}^x\partial_x(\hat{\gamma}^y\hat{\gamma}^z B_x) = \hat{\omega}\hat{\gamma}^t\partial_x B_x\,,$$
$$\hat{\gamma}_y\partial_y(\hat{\gamma}^t\hat{\gamma}^x B_x) = \hat{\omega}\hat{\gamma}^x\partial_y B_x\,.$$

Maxwell Equations

components	1	4	6	4	1
object	point	line	surface	3-volume	4-voume
line-type	$\hat{\delta}\hat{\mathcal{V}}$	$\hat{\mathcal{V}}$	$\hat{d}\hat{\mathcal{V}}$		
surface-type		$\hat{\delta}\hat{\mathcal{F}}$	$\hat{\mathcal{F}}$	$\hat{d}\hat{\mathcal{F}}$	

$$\hat{D}\hat{\mathcal{F}} = \hat{\delta}\hat{\mathcal{F}} + \hat{d}\hat{\mathcal{F}} = J_\mu \hat{\gamma}^\mu \equiv \hat{\mathcal{J}} \tag{1}$$

$$\hat{\delta}\hat{\mathcal{F}} = \hat{\mathcal{J}} \quad \longrightarrow \quad \nabla \cdot \mathbf{E} = \rho\,, \quad \partial_t \mathbf{E} + \nabla \times \mathbf{B} = \mathbf{J}$$

$$\hat{d}\hat{\mathcal{F}} = 0 \quad \longrightarrow \quad \nabla \cdot \mathbf{B} = 0\,, \quad \partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$$

$$\longrightarrow \quad \nabla \times \mathbf{A} = \mathbf{B}$$

EM Fields in Matrix Form

$$
\begin{aligned}
\hat{\mathcal{F}} =& \hat{\gamma}^t \hat{\gamma}^x E_x + \hat{\gamma}^t \hat{\gamma}^y E_x + \hat{\gamma}^t \hat{\gamma}^z E_z \\
&+ \hat{\gamma}^y \hat{\gamma}^z B_x + \hat{\gamma}^z \hat{\gamma}^x B_y + \hat{\gamma}^x \hat{\gamma}^y B_z
\end{aligned}
$$

with Dirac Basis

$$
\hat{\mathcal{F}} = \begin{pmatrix}
B_z & B_x - iB_y & iE_z & iE_x + E_y \\
B_x + iB_y & -B_z & iE_x - E_y & -iE_z \\
iE_z & iE_x + E_y & B_z & B_x - iB_y \\
iE_x - E_y & -iE_z & B_x + iB_y & -B_z
\end{pmatrix}
$$

Dirac Form

$$\hat{\mathcal{F}} = \begin{pmatrix} B_z & B_x - iB_y & iE_z & iE_x + E_y \\ B_x + iB_y & -B_z & iE_x - E_y & -iE_z \\ iE_z & iE_x + E_y & B_z & B_x - iB_y \\ iE_x - E_y & -iE_z & B_x + iB_y & B_z \end{pmatrix}$$

$$\Downarrow$$

$$\hat{\psi}_{EM} = \begin{pmatrix} B_z \\ B_x + iB_y \\ iE_z \\ iE_x - E_y \end{pmatrix}$$

Maxwell Equations in Dirac Form

Operator: Derivative

$$\hat{D} = \hat{\gamma}^t \partial_t + \hat{\gamma}^x \partial_x + \hat{\gamma}^y \partial_y + \hat{\gamma}^z \partial_z$$

Operand: Wave Fields

$$\hat{\psi}_{EM} = \begin{pmatrix} B_z \\ B_x + iB_y \\ iE_z \\ iE_x - E_y \end{pmatrix}$$

Maxwell Eqs. in Dirac Form

$$\hat{D}\hat{\psi}_{EM} = \hat{\gamma}^\mu \partial_\mu \hat{\psi}_{EM} = \hat{J}$$

Maxwell Form

Dirac Wave Function

$$\hat{\psi}_{EM} = \begin{pmatrix} B_z \\ B_x + iB_y \\ iE_z \\ iE_x - E_y \end{pmatrix} \quad \Rightarrow \quad \hat{\psi}_D = \begin{pmatrix} i\alpha_0 + \beta_z \\ \beta_x + i\beta_y \\ \alpha_\omega + i\varepsilon_z \\ i\varepsilon_x - \varepsilon_y \end{pmatrix}$$

Dirac Equation in Maxwell Form (massless)

$$\partial_t \alpha_0 + \nabla \cdot \boldsymbol{\varepsilon} = 0, \qquad \nabla \alpha_0 + \partial_t \boldsymbol{\varepsilon} - \nabla \times \boldsymbol{\beta} = 0$$
$$\partial_t \alpha_\omega + \nabla \cdot \boldsymbol{\beta} = 0, \qquad \nabla \alpha_\omega + \partial_t \boldsymbol{\beta} + \nabla \times \boldsymbol{\varepsilon} = 0$$

Wave Equations

- Operator:

$$\hat{D} = \hat{\gamma}^t \partial_t + \hat{\gamma}^x \partial_x + \hat{\gamma}^y \partial_y + \hat{\gamma}^z \partial_z$$

- Operand:

Translation: vector

- **Line**
  Klein-Gordon: $\partial_t^2 \phi - \partial_x^2 \phi - \partial_y^2 \phi - \partial_z^2 \phi$
- **Surface**
  Maxwell: $\partial_t \mathbf{B} + \nabla \times \mathbf{E}$, etc.

Rotation: rotor

- **Spin**
  Dirac: $\gamma^\mu \partial_\mu \psi_D$

Summary

- 3D static fields are good excise for 4D fields.
- The Klein-Gordon, Maxwell, and Dirac equations are expressed in similar forms.
  - Klein-Gordon, Maxwell $\Rightarrow$ Dirac Matrices
  - Dirac $\Rightarrow$ Vector Analysis
- Differential Operators are identical.
- Operand (wave field) makes difference, not Operators.

Equation	Field	Matrix	Vector Analysis
Klein-Gordon	line	$D\mathcal{V} = m\phi$	$\nabla^2\phi = m^2\phi$
Maxwell	surface	$D\mathcal{F} = \mathcal{J}$	$\partial_t \mathbf{E} + \nabla \times \mathbf{B} = J$, etc.
Dirac	rotation	$D\psi_D = m\psi_D$	$\nabla \cdot \alpha_0 + \partial_t \boldsymbol{\varepsilon} - \nabla \times \boldsymbol{\beta} = m\boldsymbol{\varepsilon}$, etc.

Geometric Algebra

Vector Calculations

- Component Calculation
- Quaternion (Hamilton)
- Exterior Algebra (Grassmann)
- Clifford Algebra (Clifford)
- Vector Analysis (Gibbs, Heaviside)
- Differential Form (Cartan)
- Geometric Algebra (Hestenes)

Algebraic Relation for Geometric Algebra

$$\hat{\gamma}^{\mu}\hat{\gamma}^{\nu} + \hat{\gamma}^{\nu}\hat{\gamma}^{\mu} = 2\hat{\eta}^{\mu\nu}$$

3D $\Longrightarrow$ 4D

grade	0	1	2	3
unit	$\hat{1}$	$\hat{\sigma}_i$	$\hat{\sigma}_i\hat{\sigma}_j$	$\hat{\sigma}_i\hat{\sigma}_j\hat{\sigma}_k$
object	point	line	surface	voume
components	1	3	3	1
elec.	$\nabla \cdot \mathbf{E}$	$\mathbf{E}$	$\nabla \times \mathbf{E}$	
mag.		$\nabla \times \mathbf{B}$	$\mathbf{B}$	$\nabla \cdot \mathbf{B}$

grade	0	1	2	3	4
unit	$\hat{1}$	$\hat{\gamma}^\mu$	$\hat{\gamma}^\mu\hat{\gamma}^\nu$	$\hat{\gamma}^\mu\hat{\gamma}^\nu\hat{\gamma}^\alpha$	$\hat{\gamma}^\mu\hat{\gamma}^\nu\hat{\gamma}^\alpha\hat{\gamma}^\beta$
object	point	line	surface	3-volume	4-voume
components	1	4	6	4	1
line-type	$\hat{\delta}\hat{\mathcal{V}}$	$\hat{\mathcal{V}}$	$\hat{d}\hat{\mathcal{V}}$		
surface-type		$\hat{\delta}\hat{\mathcal{F}}$	$\hat{\mathcal{F}}$	$\hat{d}\hat{\mathcal{F}}$	

Structured Data

- Non-structured:

```
DayOfWeek:= 3 ;
ThreeDaysLater:= DayOfWeek + 3 ;
  if(ThreeDaysLater > 7)...
```

- Structured:

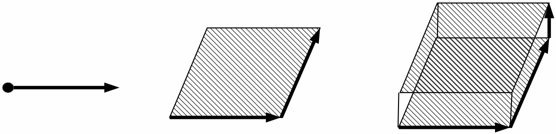
```
type DofW: {sun,mon,tue,wed,ths,fri,sat} ;
var DayOfWeek:  DofW ;

DayOfWeek:= wed ;
ThreeDaysLater:= DayOfWeek + 3 ;
```

Grassmann's father: J. G. Grassmann

Hermann Grassmann
Wikipeida より引用

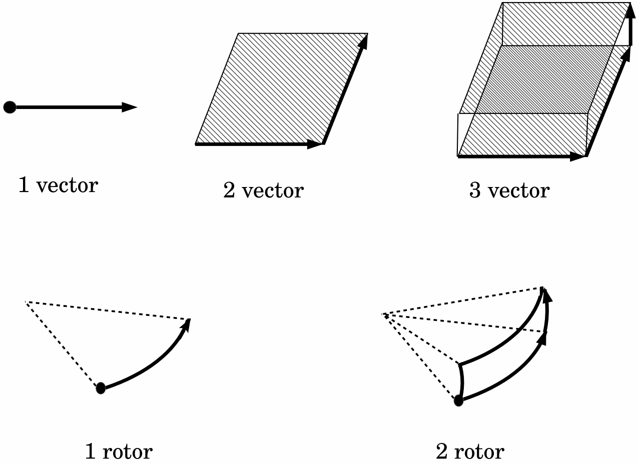
… the point is the original “producing” element; from it through construction the line emerges..., if we treat it (line) in the same manner as we formerly treated the point, then the rectangle emerges, ...

Justus G. Grassmann: “Raumlehre Theil”



u $u \wedge v$ $u \wedge v \wedge w$

Vectors vs Rotors



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