The Minimum Number of Shaded Cells in an Aqre Grid

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Abstract

For $m, n \geq 3$, consider all possible ways to shade some of the cells of a rectangular $m \times n$ grid such that (i) the shaded cells form an orthogonally connected region and (ii) every 1×4 or 4×1 rectangle in the grid contains at least one shaded cell and at least one unshaded cell. We call such shadings *Aqre grids*, after the pencil-and-paper logic puzzle genre Aqre invented by Eric Fox in October 2020. We prove that the minimum number of shaded cells in an $m \times n$ Aqre grid is between $\frac{2}{5}mn - \frac{3}{5}(m+n+1)$ and $\frac{2}{5}mn + \frac{3}{5}(m+n-3)$.

Contents

 2 The upper bound 3 The lower bound 4 Penalty theory 5 Further results and questions 	1	Introduction	2
 3 The lower bound 4 Penalty theory 5 Further results and questions 	2	The upper bound	3
4 Penalty theory5 Further results and questions	3	The lower bound	6
5 Further results and questions	4	Penalty theory	7
	5	Further results and questions	8

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Figure 1: An example Agre puzzle by Robert Vollmert and its unique solution.

1 Introduction

Aqre is a pencil-and-paper logic puzzle genre invented by Eric Fox in October 2020.¹ The solver is presented with a rectangular grid divided into boldly outlined "regions", some of which are marked with a numerical clue. An example is shown on the left side of Figure 1.

The goal of an Aqre puzzle is to shade some of the cells of the grid in such a way that

- (i) The shaded cells form a single orthogonally connected region.
- (ii) Every 1×4 or 4×1 rectangle in the grid contains at least one shaded cell and at least one unshaded cell.
- (iii) A numerical clue written in a boldly outlined region indicates the number of shaded cells in that region.

Regardless of how the grid is divided into regions and which numerical clues are present, rules (i) and (ii) constrain which cells can be shaded in a completed Aqre grid. This motivates the following definition:

Definition 1. Let $m, n \ge 1$. An $m \times n$ Aqre grid is a way of shading the cells of a rectangular $m \times n$ grid such that

- (i) The shaded cells form a single orthogonally connected region.
- (ii) Every 1×4 or 4×1 rectangle in the grid contains at least one shaded cell and at least one unshaded cell.

(Formally, an Aqre grid is a subset of $\{1, \ldots, m\} \times \{1, \ldots, n\}$, but we will have no need for that formalism in this paper.)

¹As of November 4, 2020, there have been 45 Aqre puzzles published in the puzz.link database since its introduction to the database on October 27, 2020. That is an average of 5.0 Aqre puzzles per day, making Aqre one of the most popular types in the database.

A natural question to ask is: what is the minimum number of shaded cells in an $m \times n$ Aqre grid? In this note, we will prove the following theorem, which gives effective upper and lower bounds on this minimum that differ by only O(m + n). (The *maximum* number of shaded cells in an $m \times n$ Aqre grid is much easier to compute and its computation is left to the reader.)

Theorem 1. Let f(m,n) denote the minimum number of shaded cells among all $m \times n$ Aqre grids. Then, for all $m, n \ge 3$, we have

$$\frac{2}{5}mn - \frac{3}{5}(m+n+1) \le f(m,n) \le \frac{2}{5}mn + \frac{3}{5}(m+n-3).$$

In particular, the minimum density of shaded cells in an $m \times n$ Aqre grid, which is the fraction $\frac{f(m,n)}{mn}$, approaches $\frac{2}{5}$ in the limit as $m, n \to \infty$. In Section 2, we will prove the second inequality of Theorem 1. In Section 3,

In Section 2, we will prove the second inequality of Theorem 1. In Section 3, we will prove the first inequality of Theorem 1. In Section 4, we will describe how our proof of Theorem 1 yields a "penalty theory" for Aqre, which has potential applications in both setting and solving Aqre puzzles. In Section 5, we will discuss some other results and conjectures.

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2 The upper bound

Let $m, n \geq 3$. In this section, we begin the proof of Theorem 1 by proving that

$$f(m,n) \le \frac{2}{5}mn + \frac{3}{5}(m+n-3).$$
(1)

For this, it suffices to construct an $m \times n$ Aqre grid with at most $\frac{2}{5}mn + \frac{3}{5}(m + n - 3)$ shaded cells.

Consider the infinite grid depicted in Figure 2. It is shown divided into four regions:

- (A) a 3×3 square in the upper left,
- (B) an infinite horizontal strip of height 3 in the upper right,
- (C) an infinite vertical strip of width 3 in the lower left, and
- (D) an infinite quadrant in the lower right.



Figure 2: An infinite Aqre grid with a shaded cell density of $\frac{2}{5}$. Truncating this infinite grid to an $m \times n$ rectangle yields an $m \times n$ Aqre grid.

Within each of the regions (B), (C), and (D), the shading is periodic with a period of 5.

Let R be the $m \times n$ rectangle that shares its upper left corner with the infinite grid depicted in Figure 2. Observe that R is an $m \times n$ Aqre grid. Hence, it remains to show that the number of shaded cells in R is at most $\frac{2}{5}mn + \frac{3}{5}(m+n-3)$. Let us now count them, region by region.

- (A) Since $m, n \ge 3$, the entirety of region (A) is contained within the rectangle R. So there are exactly $\boxed{3}$ shaded cells that are in both (A) and R.
- (B) Since every 3×5 rectangle in region (B) contains exactly 9 shaded cells and the shading in region (B) is periodic with period 5, the number of shaded cells that are in both (B) and R is precisely

$$\begin{cases} \frac{9}{5}(n-3) & \text{if } n-3 \equiv 0 \pmod{5} \\ \frac{9}{5}(n-4)+3 & \text{if } n-3 \equiv 1 \pmod{5} \\ \frac{9}{5}(n-5)+4 & \text{if } n-3 \equiv 2 \pmod{5} \\ \frac{9}{5}(n-6)+6 & \text{if } n-3 \equiv 3 \pmod{5} \\ \frac{9}{5}(n-7)+8 & \text{if } n-3 \equiv 4 \pmod{5} \end{cases}$$

This is less than or equal to $\frac{9}{5}n - \frac{21}{5}$ in all cases.

- (C) Similarly, the number of shaded cells that are in both (C) and R is at most $\boxed{\frac{9}{5}m \frac{21}{5}}$.
- (D) Let R' be the intersection of R with region (D), which is an $(m-3) \times (n-3)$ rectangle. Let x and y be the remainder when m-3 and n-3 are divided by 5, respectively, and let S be the $x \times y$ rectangle that shares its upper left corner with region (D).

Observe that S is contained in R', and that $R' \setminus S$ can be tiled by 1×5 and 5×1 rectangles. Also observe that within region (D), each 1×5 or 5×1 rectangle contains exactly 2 shaded cells. Hence, exactly a $\frac{2}{5}$ fraction of $R' \setminus S$ is shaded.

On the other hand, it is easy to check by considering all possible values of x and y that at most a $\frac{2}{5}$ fraction of S is shaded. Hence, at most a $\frac{2}{5}$ fraction of R' is shaded. That is, the number of shaded cells that are in both (D) and R is at most $\boxed{\frac{2}{5}(m-3)(n-3)}$.

Hence the total number of shaded cells in R is at most

$$3 + \left(\frac{9}{5}n - \frac{21}{5}\right) + \left(\frac{9}{5}m - \frac{21}{5}\right) + \frac{2}{5}(m-3)(n-3) = \frac{2}{5}mn + \frac{3}{5}(m+n-3).$$

Since R is an Aqre grid, the inequality (1) follows.



Figure 3: A 7×9 Aqre grid with 31 shaded cells.

3 The lower bound

Let $m, n \geq 3$. In this section, we will prove that

$$f(m,n) \ge \frac{2}{5}mn - \frac{3}{5}(m+n+1),$$

completing the proof of Theorem 1.

Consider any $m \times n$ Aqre grid and let s be the number of shaded cells. We wish to show that $s \ge \frac{2}{5}mn - \frac{3}{5}(m+n+1)$. We will use the grid in Figure 3 as an example, with m = 7, n = 9, and s = 31.

Lemma 1. Let h be the number of pairs of horizontally adjacent shaded cells. For example, in Figure 3, the pairs of horizontally adjacent shaded cells are marked by white circles and h = 16. Then $4s - 3h \ge m(n-3)$.

Proof. For i = 1, ..., m, let s_i be the number of shaded cells in row i and let h_i be the number of pairs of horizontally adjacent shaded cells in row i. For example, in Figure 3, we have $(s_1, ..., s_7) = (5, 6, 4, 2, 6, 5, 3)$ and $(h_1, ..., h_7) = (2, 4, 2, 0, 4, 3, 1)$. Clearly $s = s_1 + \cdots + s_m$ and $h = h_1 + \cdots + h_m$.

For i = 1, ..., m, the difference $s_i - h_i$ is equal to the number of runs of shaded cells in row i. Hence, the number of runs of unshaded cells in row i is at most $s_i - h_i + 1$. Since a run of unshaded cells cannot have any more than three unshaded cells, the number of unshaded cells in row i is at most $3(s_i - h_i + 1)$. On the other hand, the number of unshaded cells in row i is equal to $n - s_i$. We conclude

$$n - s_i \le 3(s_i - h_i + 1)$$

which can be rearranged to

$$4s_i - 3h_i \ge n - 3. \tag{2}$$

Adding the inequalities (2) for i = 1, ..., m yields the desired result.

Let *h* be the number of pairs of horizontally adjacent shaded cells and let v be the number of pairs of vertically adjacent shaded cells. For example, in Figure 3 we have v = 14. Then by Lemma 1 we have $4s - 3h \ge m(n-3)$, and symmetrically we have $4s - 3v \ge n(m-3)$. Adding these inequalities, we conclude that

$$8s - 3(h + v) \ge m(n - 3) + n(m - 3).$$
(3)

Consider the graph Γ whose vertices are the shaded cells, and which has one edge joining each pair of orthogonally connected shaded cells. The graph Γ is connected, with s vertices and h + v edges. Hence

$$h + v \ge s - 1,\tag{4}$$

with equality if and only if Γ is a tree.

Combining (3) and (4) yields

$$8s \ge m(n-3) + n(m-3) + 3(s-1),$$

 \mathbf{SO}

$$s \ge \frac{2}{5}mn - \frac{3}{5}(m+n+1),$$

as desired. Together with the results of Section 2, this completes the proof of Theorem 1. $\hfill \Box$

Remark 1. In the proof that $s \geq \frac{2}{5}mn - \frac{3}{5}(m+n+1)$, we never used the fact that there is no 1×4 or 4×1 rectangle of shaded cells. However, it does appear that this rule has an impact on the minimum number of shaded cells.

4 Penalty theory

Carefully examining the proof in Section 3 and quantifying the degree to which the combined inequalities (2) and (4) fail to hold with equality yields the following result, which can be thought of as a "penalty theory" for Aqre.

Theorem 2. In an $m \times n$ Aqre grid,

- let r_1 be the number of horizontal or vertical runs of exactly one unshaded cell (counting an isolated unshaded cell twice: once in its row and once in its column),
- let r_2 be the number of horizontal or vertical runs of exactly two unshaded cells,
- let s_e be the number of shaded cells on the edge of the grid, but not in the corner,
- let s_c be the number of shaded cells in a corner of the grid, and

• let g = a - s + 1, where a is the number of pairs of orthogonally adjacent shaded cells. (Note that if we draw a segment joining the centers of any two orthogonally adjacent cells, then g is equal to the number of bounded regions formed by the drawn segments. In particular, g is greater than or equal to the number of fully shaded 2×2 squares.)

Let $P = 2r_1 + r_2 + 3s_e + 6s_c + 3g$ (the "penalty"). Then the number of shaded cells in the grid is exactly

$$\frac{2}{5}mn - \frac{3}{5}(m+n+1) + \frac{P}{5}.$$

It is possible to use Theorem 2 to improve the lower bound of Theorem 1 by $\frac{2}{5}(m+n) + O(1)$, but we will not do so in this paper.

5 Further results and questions

Improving the construction

Can the construction of Section 2 can be improved? We conjecture that it cannot be improved by more than a constant number of shaded cells.

Conjecture 1. There exists a constant C such that

$$f(m,n) \ge \frac{2}{5}mn + \frac{3}{5}(m+n-3) - C$$

for all $m, n \geq 3$.

Changing the number 4 in the Aqre rules

Suppose we change rule (ii) of Aqre to:

(ii) Every $1 \times b$ or $b \times 1$ rectangle in the grid contains at least one shaded cell and at least one unshaded cell.

for some $b \ge 4$. It seems that the methods in this paper can be used to show that the minimum density of shaded cells then approaches $\frac{2}{b+1}$ in the limit. Can this be made precise?